Mastermath	Commutative algebra exam	14:00-17:00 29/1/2025

- Write your name, university, and student number on every sheet you hand in.
- You may not use any books or notes, or a calculator, during the exam.
- Unless stated otherwise, you need to give full proofs in all your answers. You are allowed to use results that are treated in the book and lectures.
- If you cannot do a part of a question, you may still use its conclusion later on.
- There are **5** questions in total. The exam continues on the back of this sheet.

(1) At the top of the following table, two ring homomorphisms $f\colon R \to A$ are listed; in the first column, $S = \{2^n : n \in \mathbb{N}\}$.

	$f\colon \mathbb{Z} \to S^{-1}\mathbb{Z}$	$f\colon \mathbb{Q}[t] \to \mathbb{Q}[x,y]/(xy); t \mapsto x$
A is finitely generated as R-algebra	(b)	
A is finitely generated as R-module	(c)	
f is flat		

(a) **Copy the above table onto your answer paper**, then fill in each box in the table with T or F, according to whether or not the given property is true for the given ring homomorphism f in that column. **You do not need to justify your answers to this part.**

(b) Prove your answer in the box marked (b).

(c) Prove your answer in the box marked (c).

(2) Let $R = \mathbb{Q}[x,y]$ and $A = R/(y^2 - x^3)$. You can use without proof the fact that A is an integral domain of dimension 1.

(a) Let $I = (x^3, xy^2) \subseteq R$. Compute a minimal primary decomposition of I and its isolated primes.

(b) Show that $A_{(x-1,y-1)}$ is a DVR.

(c) Is A a Dedekind domain? Justify your answer. *Hint: look at $A_{(x,y)}$.*

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(3) A map of rings $\varphi : R \to A$ is called **universally injective** if for every R-module M, the natural map
$$R \otimes_R M \to A \otimes_R M$$
is injective.

(a) Give an example of a ring homomorphism which is injective but not universally injective.

(b) Let $f \colon R \to B$ be a ring homomorphism. Suppose that φ is universally injective. Show that the natural map $B \to A \otimes_R B$ is universally injective.

(c) Show that, if φ is universally injective, then the induced map on spectra
$$\varphi^* : \operatorname{Spec} A \to \operatorname{Spec} R$$
is surjective. *Hint: it may help to take M to be a quotient of a localisation of R.*

(d) Give an example of a non-injective ring homomorphism $\varphi : R \to A$ such that $\varphi^* : \operatorname{Spec} A \to \operatorname{Spec} R$ is surjective.

(e) Show that, if φ is flat and $\varphi^* : \operatorname{Spec} A \to \operatorname{Spec} R$ is surjective, then φ is universally injective. *Hint: Suppose φ is not universally injective, and try to find a proper ideal I of R such that A/IA is zero. What happens if $\mathfrak{m}$ is a maximal ideal of R containing I?*

(4) Let $R = \mathbb{Q}[x, y, z]/(x^3, y^6)$. Let $I = (x, z)$ and $J = (z)$. Denote by R^I (resp. R^J) the completion of R with respect to I (resp. J).

(a) Is R complete with respect to (x, y)? Justify your answer.

(b) Is there an isomorphism of rings between R^I and R^J? Justify your answer.

(c) Is the image of $z + 3$ in R^J a unit of R^J? Justify your answer.

(d) Is R complete with respect to (x, z)? Justify your answer.

(e) Is R an Artin ring?

(5) Let k be a field. Let $A = k[x, y, z]/(x^3, yz)$ and consider the ideal $\mathfrak{p} = (x, y)$ of A. Let $B = A_{\mathfrak{p}}$ and $\mathfrak{q} = \mathfrak{p}B$.

(a) Show that $B \cong k[x, z]_{(x)}/(x^3)$.

Let $G_{\mathfrak{q}}(B) = \bigoplus_{n \geq 0} \mathfrak{q}^n/\mathfrak{q}^{n+1}$.

(b) Compute the Poincaré series of $G_{\mathfrak{q}}(B)$ as a rational function in a variable t. Take the additive function λ to be the length as $B/\mathfrak{q}$-modules.

(c) Show that $G_{\mathfrak{q}}(B)$ is Artinian.

(d) Compute the dimension of A as a ring.